International Baccalaureate®
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MARKSCHEME

May 2012

FURTHER MATHEMATICS

Standard Level

Paper 2

14 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2012**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.

- If a part is completely wrong, stamp ***A0*** by the final answer.

- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

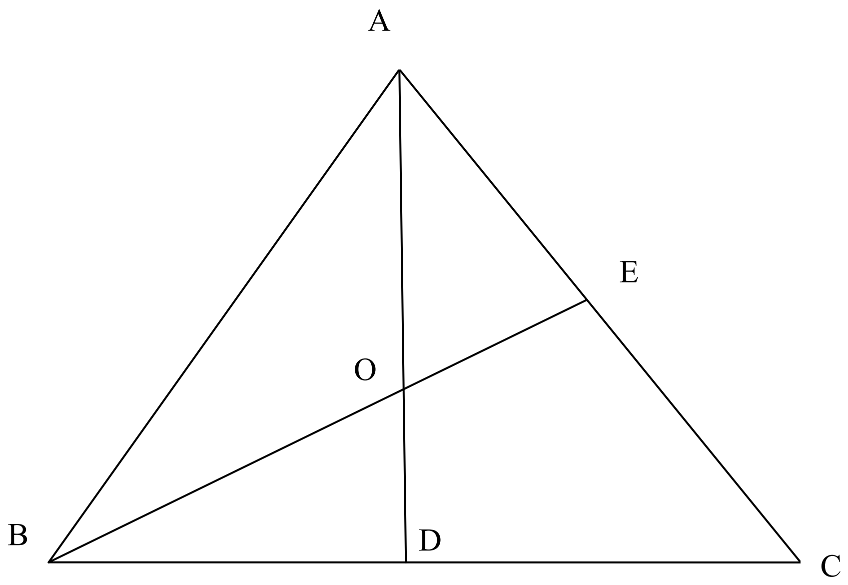
A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. Part A



(a) consider the above diagram – [AD] and [BE] are the medians and O is therefore both the incentre and the circumcentre *(R1)*

let $\text{AB} = d$ and let R denote the radius of the circumcircle

then,

$R = \text{AO} = \text{AE}\sec 30^\circ$ ***M1***

$= \frac{d}{2} \times \frac{2}{\sqrt{3}} = \frac{d}{\sqrt{3}}$ ***(A1)***

area of circumcircle $= \pi R^2 = \frac{\pi d^2}{3}$ ***A1***

area of triangle $= \frac{1}{2}\text{AB.AC}\sin \text{BAC}$ ***M1***

$= \frac{\sqrt{3}d^2}{4}$ ***(A1)***

$\frac{\sqrt{3}d^2}{4} = 1 \Rightarrow d^2 = \frac{4}{\sqrt{3}}$ ***A1***

area of circumcircle $= \frac{4\pi}{3\sqrt{3}}$ (2.42) ***A1***

[8 marks]

continued …

Question 1 continued

(b) let r denote the radius of the incircle
then

$r = \text{OE} = \text{AE}\tan 30^{\circ}$ ***M1***

$= \dfrac{d}{2\sqrt{3}}$ ***(A1)***

area of incircle $= \pi r^2 = \dfrac{\pi d^2}{12}$

$= \dfrac{\pi}{3\sqrt{3}}$ (0.605) ***A1***

[3 marks]

Part B

(a) $\text{AP}^2 = (x-1)^2 + y^2$ and $\text{BP}^2 = x^2 + (y-1)^2$ ***A1***

$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$ ***M1***

$y = x$ which is the equation of a straight line ***A1***

[3 marks]

(b) (i) $x^2 - 2x + 1 + y^2 = k^2(x^2 + y^2 - 2y + 1)$ ***M1***

$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 2x - 2k^2 y + k^2 - 1 = 0$ ***A1***

$x^2 + y^2 + \dfrac{2x}{k^2 - 1} - \dfrac{2k^2 y}{k^2 - 1} + 1 = 0$ ***A1***

by completing the squares or quoting the standard result, ***M1***
coordinates of C are

$\left(-\dfrac{1}{k^2 - 1}, \dfrac{k^2}{k^2 - 1}\right)$ ***A1***

(ii) let (x, y) be the coordinates of C

attempting to find k or k^2, ***(M1)***

$k^2 = 1 - \dfrac{1}{x}$ ***(A1)***

$y = \dfrac{1 - \frac{1}{x}}{-\frac{1}{x}}$ ***(M1)***

$y = 1 - x$ ***A1***

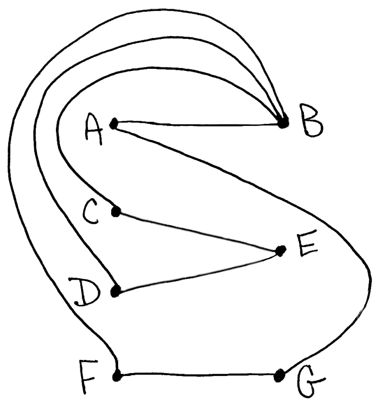
[9 marks]

Total [23 marks]

2. Part A

(a) (i) using any method, *(M1)*
find that $\{A, C, D, F\}$ and $\{B, E, G\}$ are disjoint sets of vertices *A1*
so that H is bipartite *AG*

(ii)



A1

[3 marks]

(b) (i) all vertices are of even degree *A1*
(ii) DECBAGFBD *A2*

[3 marks]

(c) (i)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^5$$ *M1*

number of walks $= 36$ *A1*

(ii) recognition of the need to find walks of length 2 and walks of length 3 *(M1)*
number of walks of length 2 from A to F $= 2$ *A1*
number of walks of length 3 from F to B $= 6$ *A1*
required number of walks $= 12$ *A1*

[6 marks]

(d) for a simple, bipartite graph to be planar,
$e \le 2v - 4 = 10$ *M1*
at the moment, $e = 8$ which means that we cannot add more than 2 edges *A1*
we see that we can add 2 edges, *e.g.* EA and EF *A1*
the maximum number of edges we can add is therefore 2 *A1*

[4 marks]

continued …

Question 2 continued

Part B

(a) evaluating successive powers of 3 *(M1)*

$3^1 \equiv 3(\text{mod}\,22),\ 3^2 \equiv 9(\text{mod}\,22),\ 3^3 \equiv 5(\text{mod}\,22)$

$3^4 \equiv 15(\text{mod}\,22),\ 3^5 \equiv 1(\text{mod}\,22)$ so $m = 5$ ***A1***

[2 marks]

(b) since $3^5 \equiv 1(\text{mod}\,22),\ 3^{45} = (3^5)^9 \equiv 1(\text{mod}\,22)$ ***M1A1***

consider $3^{49} = 3^{45} \times 3^4 \equiv 1 \times 15(\text{mod}\,22)$ so $n = 15$ ***M1A1***

[4 marks]

(c) from (a), $x = 3$ is a solution ***A1***

since $3^5 \equiv 1(\text{mod}\,22)$, the complete solution is $x = 3 + 5N$ where $N \in \bullet$ ***(M1)A1***

[3 marks]

Total [25 marks]

3. (a) (i) $\frac{\text{d}}{\text{d}\theta}\left(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right)$

$= \sec^3\theta + \sec\theta\tan^2\theta + \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}$ ***M1A1A1***

Note: Award ***M1*** for a valid attempt to differentiate either term.

$= \sec^3\theta + \sec\theta(\sec^2\theta - 1) + \sec\theta$ ***A1***

$= 2\sec^3\theta$ ***AG***

(ii) $\int \sec^3\theta \,\text{d}\theta = \frac{1}{2}\left(\sec\theta\tan + \ln(\sec\theta + \tan\theta)\right)(+\text{C})$ ***A1***

[5 marks]

(b) (i) $\frac{\text{d}y}{\text{d}x} = 1 - \frac{xy}{1 + x^2}$ ***A1***

x	y	dy/dx	0.1dy/dx	
0	1	1	0.1	***M1A1***
0.1	1.1	0.8910891089	0.08910891089	***A1***
0.2	1.189108911	0.7713252094	0.07713252094	***A1***
0.3	1.266241432			***A1***

Note: Accept tabular values correct to 3 significant figures.

$y \approx 1.27$ when $x = 0.3$ ***A1***

continued ...

Question 3 continued

(ii) consider the equation in the form

$$\frac{dy}{dx}+\frac{xy}{1+x^2}=1$$ ***(M1)***

the integrating factor I is given by

$$I=\exp\int\left(\frac{x}{1+x^2}\right)dx$$ ***A1***

$$=\exp\left(\frac{1}{2}\ln(1+x^2)\right)$$ ***A1***

$$=\sqrt{1+x^2}$$ ***A1***

Note: Accept also the fact that the integrating factor for the original equation is $\frac{1}{\sqrt{1+x^2}}$.

(iii) consider the equation in the form

$$\sqrt{1+x^2}\frac{dy}{dx}+\frac{xy}{\sqrt{1+x^2}}=\sqrt{1+x^2}$$ ***(M1)***

integrating,

$$y\sqrt{1+x^2}=\int\sqrt{1+x^2}\,dx$$ ***A1***

to integrate the right hand side, put $x=\tan\theta$, $dx=\sec^2\theta d\theta$ ***M1A1***

$$\int\sqrt{1+x^2}dx=\int\sqrt{1+\tan^2\theta}\ .\sec^2\theta d\theta$$ ***A1***

$$=\int\sec^3\theta d\theta$$ ***A1***

$$=\frac{1}{2}\left(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\right)$$

$$=\frac{1}{2}\left(x\sqrt{1+x^2}+\ln\left(x+\sqrt{1+x^2}\right)\right)$$ ***A1***

the solution to the differential equation is therefore

$$y\sqrt{1+x^2}=\frac{1}{2}\left(x\sqrt{1+x^2}+\ln\left(x+\sqrt{1+x^2}\right)\right)+C$$ ***A1***

Note: Do not penalize the omission of C at this stage.

$y=1$ when $x=0$ gives $C=1$ ***M1A1***

the solution is $y=\frac{1}{2\sqrt{1+x^2}}\left(x\sqrt{1+x^2}+\ln\left(x+\sqrt{1+x^2}\right)\right)+\frac{1}{\sqrt{1+x^2}}$ ***A1***

(iv) when $x=0.3$, $y=1.249\ldots$ ***A1***

the approximation is only correct to 1 significant figure ***A1***

[24 marks]

Total [29 marks]

4. Part A

(a) recognising that the function needs to be injective and surjective ***R1***

Note: Award ***R1*** if this is seen anywhere in the solution.

injective:
let $\boldsymbol{U}, \boldsymbol{V} \in$° ×° be 2-D column vectors such that $\boldsymbol{AU} = \boldsymbol{AV}$ ***M1***
$\boldsymbol{A}^{-1}\boldsymbol{AU} = \boldsymbol{A}^{-1}\boldsymbol{AV}$ ***M1***
$\boldsymbol{U} = \boldsymbol{V}$ ***A1***
this shows that f is injective
surjective:
let $\boldsymbol{W} \in$° ×° ***M1***
then there exists $\boldsymbol{Z} = \boldsymbol{A}^{-1}\boldsymbol{W} \in$° ×° such that $\boldsymbol{AZ} = \boldsymbol{W}$ ***M1A1***
this shows that f is surjective
therefore f is a bijection ***AG***

[7 marks]

(b) (i) the relationship is $ad = bc$ ***A1***

(ii) it follows that $\frac{c}{a} = \frac{d}{b} = \lambda$ so that $(c, d) = \lambda(a, b)$ ***A1***

(iii) **EITHER**

let $\boldsymbol{W} = \begin{bmatrix} p \\ q \end{bmatrix}$ be a 2-D vector

then $\boldsymbol{AW} = \begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$ ***M1***

$= \begin{bmatrix} ap + bq \\ \lambda(ap + bq) \end{bmatrix}$ ***A1***

the image always satisfies $y = \lambda x$ so f is not surjective and therefore not a bijection ***R1***

OR

consider

$\begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} ab \\ \lambda ab \end{bmatrix}$ ***M1***

$\begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix}\begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} ab \\ \lambda ab \end{bmatrix}$ ***A1***

this shows that f is not injective and therefore not a bijection ***R1***

[5 marks]

continued ...

Question 4 continued

Part B

(a) the identity element is 0 ***R1***
consider, for $1 \le r \le m$,
using 1 as a generator ***M1***
1 combined with itself r times gives r and as r increases from 1 to m, the group is generated ending with 0 when $r = m$ ***A1***
it is therefore cyclic ***AG***

[3 marks]

(b) (i) by Lagrange the order of each element must be a factor of m and if m is prime, its only factors are 1 and m ***R1***
since 0 is the only element of order 1, all other elements are of order m and are therefore generators ***R1***

(ii) since $x +_m (m - x) = 0$, ***(M1)***
the inverse of x is $(m - x)$ ***A1***

(iii) consider

element	inverse
1	$m-1$
2	$m-2$
.	.
.	.
.	.
$\frac{1}{2}(m-1)$	$\frac{1}{2}(m+1)$

M1A1

there are $\frac{1}{2}(m-1)$ inverse pairs ***A1 N1***

> **Note**: Award ***M1*** for an attempt to list the inverse pairs, ***A1*** for completing it correctly and ***A1*** for the final answer.

[7 marks]

(c) since a, b are unequal primes the only factors of m are a and b
there are therefore only subgroups of order a and b ***R1***
they are
$\{0, a, 2a, \ldots, (b-1)a\}$ ***A1***
$\{0, b, 2b, \ldots, (a-1)b\}$ ***A1***

[3 marks]

Total [25 marks]

5. (a) $E(X)=\int_a^b xf(x)\,\mathrm{d}x$ ***M1***

$=[xF(x)]_a^b-\int_a^b F(x)\,\mathrm{d}x$ ***A1***

$=bF(b)-aF(a)-\int_a^b F(x)\,\mathrm{d}x$ ***A1***

$=b-\int_a^b F(x)\,\mathrm{d}x$ because $F(a)=0$ and $F(b)=1$ ***A1***

[4 marks]

(b) (i) let G denote the cumulative distribution function of Y

$G(y)=\int_0^y \cos t\,\mathrm{d}t$ ***M1***

$=\left[\sin t\right]_0^y$ ***(A1)***

$=\sin y$ ***A1***

$E(Y)=\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}\sin y\,\mathrm{d}y$ ***M1***

$=\frac{\pi}{2}+\left[\cos y\right]_0^{\frac{\pi}{2}}$ ***A1***

$=\frac{\pi}{2}-1$ ***A1***

(ii) CDF of $U=P(U\le u)$ ***M1***

$=P(Y^n\le u)$ ***A1***

$=P(Y\le u^{\frac{1}{n}})$ ***A1***

$=G(u^{\frac{1}{n}})$ ***(A1)***

$=\sin\left(u^{\frac{1}{n}}\right)$ ***A1***

(iii) m_y satisfies the equation $\sin m_y=\frac{1}{2}$ ***A1***

m_u satisfies the equation $\sin\left(m_u^{\frac{1}{n}}\right)=\frac{1}{2}$ ***A1***

therefore $m_y=m_u^{\frac{1}{n}}$ ***A1***

$m_u=m_y^n$ ***AG***

[14 marks]

Total [18 marks]